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LOAD TRANSFER AND SURFACE WAVE PROPAGATION IN FIBER REINFORCED COMPOSITE MATERIALS

M. JAHANMIR† and H. F. TIERSTEN

Department of Mechanical Engineering, Aeronautical Engineering and Mechanics, Rensselaer Polytechnic Institute, Troy, NY 12181, U.S.A.

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Abstract—A system of linear elastic equations recently obtained for fiber reinforced composite materials is applied to some simple problems concerning the transfer of load from the reinforcement to the matrix. The same equations are applied to surface waves propagating in the direction of the fiber reinforcement. Since the defined constants occurring in the above-mentioned linear equations for a two-constituent composite material have never been measured, calculations cannot be performed. When the model is simplified sufficiently, the effective constants in the description can be partially estimated from the known elastic constants of the individual constituents in the composite. With the reduced equations calculations are performed for surface waves propagating both in and normal to the direction of the fiber reinforcement. The calculations indicate the existence of a high (optical type) as well as a low (acoustic type) surface wave mode, both of which are dispersive. We believe the optical type mode is an analytical consequence of the simplified model and does not actually exist. The dispersion of the acoustic type surface wave mode could provide a means of non-destructively evaluating the integrity of a fiber reinforced composite material.

1. INTRODUCTION

In a recent investigation[1] a system of linear elastic equations for a two-constituent composite material was obtained from a general nonlinear system for N-constituents. The general system of nonlinear equations was obtained from a model of the composite consisting of interpenetrating solid continua, in which the motion of a point of the combined continuum could be finite while the relative motion of each of the constituents was constrained to be infinitesimal in order that the solid composite not rupture. The aforementioned linear equations were written in explicit detail for the isotropic and transversely isotropic symmetries. The linear elastic equations for the two-constituent transversely isotropic composite obtained in the earlier work[1] form the basis of the work presented here.

In this paper the aforementioned linear equations for the transversely isotropic composite material are applied in the analysis of some simple but very interesting one-dimensional static load transfer problems and the propagation of straight-crested surface waves. In the two particular one-dimensional static problems considered, axial loading is applied to the fiber reinforcement which enters the unloaded matrix. In one case both the matrix and reinforcement are held fixed at the supporting end, while in the other case the reinforcement ends at some distance into the matrix. In both cases the influence of gravity is included in the analysis. In each instance the rate of transfer of stress from the reinforcement to the matrix is determined in terms of the defined material coefficients of the two-constituent composite as a byproduct of the solution of a simple system of ordinary differential equations with constant coefficients. The treatment of such problems within the framework of the theory of linear elasticity is prohibitively complicated and such problems cannot even be mathematically defined using the ordinary strength of materials approach.

In the case of surface wave propagation only the placement of fiber reinforcement parallel to the free surface is considered. The formal solution for surface wave propagation in the direction of the fiber reinforcement is presented. However, since the material coefficients occurring in the theory have never been measured for any composite, a calculation cannot be performed. Nevertheless, if the resulting theory is reduced by making the plausible simplifying physical assumptions, for certain cases of interest, that only axial stress exists in the reinforcement and the entire interaction depends only on the relative displacements of the

†Present address: Department of Mechanical Engineering, Arya-Mehr University of Technology, P.O. Box 3406, Tehran, Iran.

constituents in the model, the remaining unknown constants in the description can be partially estimated from the known ordinary elastic constants. Then by making an additional assumption, calculations can be performed. At this point it should be noted that when the aforementioned simplification is made, the resulting linear equations are identical with an earlier system due to Bedford and Stern [2, 3]. The aforementioned partial estimation of the unknown constants in the simplified description is made using a procedure due to Martin, Bedford and Stern [4]. With the material constants thus determined the dispersive surface wave velocity has been calculated for a glass fiber reinforced phenolic resin. In this simplified theory upper (in frequency) optical type surface wave branches are found in addition to the lower acoustic type surface wave branches. It should be mentioned that it is felt that the upper surface wave branch that occurs in the simplified description will not occur in the full description because in the latter case all the independent solutions of the differential equations that remain coupled by the boundary conditions will probably not decay with depth. In each case treated the acoustic type surface wave branch turns out to be asymptotic to the non-dispersive surface wave velocity of the matrix at very long wavelengths. It should be noted that the theory employed and, of course, solutions presented are valid only for wavelengths long compared to the spacing of the fiber reinforcement in much the same manner that the theory of elasticity is valid only for wavelengths large compared to a lattice spacing. However, because of the behavior of the branches, we obtain and present results considerably beyond the range of validity of the theory. Nevertheless, we indicate the limit of validity of the theory on each branch plotted.

2. ONE-DIMENSIONAL STATIC PROBLEMS

In this section we consider some simple but interesting one-dimensional static problems for two-constituent transversely isotropic composite materials. In each instance the load is applied in the preferred direction of transverse isotropy, which lies along the length of the parallel fibers, and it is assumed that all displacement and relative displacement components transverse to this direction are constrained to vanish and that the remaining displacement variables are independent of the transverse coordinates. We consider only problems that satisfy these criteria. Under these circumstances the nontrivial linear constitutive equations take the form [5]

$$K_{11} = K_{22} = \hat{c}_2 u_{3,3} + \hat{\beta}_3 w^{(1)}_{3,3}, \tag{2.1}$$

$$K_{33} = \hat{c}_5 u_{3,3} + \hat{\beta}_6 w^{(1)}_{3,3}, \tag{2.2}$$

$$\mathcal{D}_{11} = \mathcal{D}_{22} = \hat{\beta}_5 u_{3,3} + \hat{b}_3 w^{(1)}_{3,3}, \tag{2.3}$$

$$\mathcal{D}_{33} = \hat{\beta}_6 u_{3,3} + \hat{b}_5 w^{(1)}_{3,3}, \tag{2.4}$$

$$\mathcal{F}_3 = -\hat{a}_2 w^{(1)}_3, \tag{2.5}$$

where x_3 is the preferred direction of transverse isotropy, u_3 is the non-zero displacement component of the center of mass of the combined two-constituent composite material and w_3[1] is the non-zero component of the relative displacement of the continuum representing the matrix. We employ Cartesian tensor notation and we have introduced the convention that a comma followed by an index denotes partial differentiation with respect to a space coordinate. The K_{LM} represent the components of the stress tensor for the combined continuum and the $\mathcal{D}_{LM}$ represent the relative stress tensor which is defined by

$$\mathcal{D}_{LM} = \tau^{(1)}_{LM} - r\tau^{(2)}_{LM}, \quad r = \rho^{(1)}/\rho^{(2)}, \quad \rho = \rho^{(1)} + \rho^{(2)}, \tag{2.6}$$

where $\tau^{(m)}_{LM}$ and $\rho^{(m)}$ represent the components of the stress tensor and mass density, respectively, of each of the interpenetrating continua. The vector field $\mathcal{F}_M$ is related to the volumetric force of interaction between the two constituents by the relation

$$\mathcal{F}_M = {}^L F^{12}_M (1 + r), \tag{2.7}$$

where ${}^L F^{12}_M$ is the volumetric force exerted by continuum 2 on continuum 1. At this point it is to be noted that the $\tau^{(m)}_{LM}$ and $\rho^{(m)}$ do not represent the actual components of stress and mass density of each of the constituents in the composite, but only represent those quantities in each of the interpenetrating continua, which occupy the same region of space and, respectively, represent each constituent in the *model*. As a consequence, if A^m and A^f represent the areas occupied by the matrix and fibers, respectively, in a typical area A of the interpenetrating continua normal to the fiber length, we have

$$A = A^m + A^f,\ \rho^{(1)} = \rho^m A^m/A,\ \rho^{(2)} = \rho^f A^f/A,$$
$$\tau_{ij}^m = \tau_{ij}^{(1)} A/A^m,\ \tau_{ij}^f = \tau_{ij}^{(2)} A/A^f, \tag{2.8}$$

where the variables with the superscripts m and f represent the actual respective quantities in the matrix and fiber reinforcement, respectively. The remaining nontrivial stress equations of equilibrium and relative stress equations of equilibrium are

$$K_{33,3} + \rho f_3 = 0, \tag{2.9}$$

$$\mathcal{D}_{33,3} + \mathcal{F}_3 + \rho^{(1)}\tilde{f}_3 = 0, \tag{2.10}$$

where

$$\rho f_3 = \rho^{(1)} f_3^{(1)} + \rho^{(2)} f_3^{(2)},\ \tilde{f}_3 = f_3^{(1)} - f_3^{(2)}, \tag{2.11}$$

and $f_3^{(1)}$ and $f_3^{(2)}$ denote the components of body force per unit mass in the continua representing the matrix and fiber reinforcement, respectively, which in the case of the gravity force are the same as the body force intensities f_3^m in the matrix and f_3^f in the fiber reinforcement, both of which equal g.

The substitution of (2.2), (2.4), (2.5) and (2.11) into (2.9) and (2.10) yields

$$\hat{c}_5 u_{3,33} + \hat{\beta}_6 w_{3,33}^{(1)} + \rho g = 0, \tag{2.12}$$

$$\hat{\beta}_6 u_{3,33} + \hat{b}_5 w_{3,33}^{(1)} - \hat{a}_2 w_3^{(1)} = 0, \tag{2.13}$$

which are the one-dimensional displacement equations of equilibrium that apply to the one-dimensional static problems treated in this section. The solution to (2.12) and (2.13) may be written in the form

$$u_3 = -\frac{\hat{\beta}_6}{\hat{c}_5}(A e^{-\alpha x_3} + D e^{\alpha x_3}) + \frac{1}{2\hat{c}_5}\rho g x_3^2 + B x_3 + C$$
$$w_3^{(1)} = A e^{-\alpha x_3} + D e^{\alpha x_3} + \gamma g, \tag{2.14}$$

where

$$\alpha^2 = \hat{c}_5 \hat{a}_2/(\hat{c}_5 \hat{b}_5 - \hat{\beta}_6^2),\ \gamma = \rho\hat{\beta}_6/\hat{c}_5\hat{a}_2, \tag{2.15}$$

and A, B, C and D are arbitrary constants, to be found by satisfying boundary conditions in a given one-dimensional problem.

In the first specific problem we consider, a total compressive force P is applied to all the fibers crossing a given cross-sectional area of a combined composite material body of height l resting on a rigid surface, as shown in Fig. 1. Since the supporting surface is rigid, there is no displacement in either the matrix or fibers at the support and we have

$$u_3 + w_3^{(1)} = 0,\ u_3 + w_3^{(2)} = 0,\ \text{at } x_3 = 0. \tag{2.16}$$

Since u_k is defined as the displacement of the center of mass of a point of the combined composite continuum, we have

$$\rho^{(1)} w_3^{(1)} + \rho^{(2)} w_3^{(2)} = 0, \tag{2.17}$$

which with (2.16) enables us to write the boundary conditions

$$u_3 = 0,\ w_3^{(1)} = 0,\ \text{at } x_3 = 0. \tag{2.18}$$

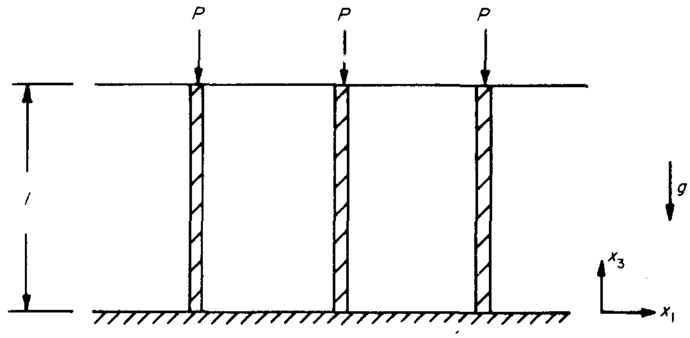

Fig. 1. Schematic diagram of loaded fiber reinforced composite supported on a rigid base.

Since no force is applied to the matrix at $x_3 = l$, we have

$$\tau_{33}^{(1)} = 0,\ \tau_{33}^{(2)} = -p_0,\ \text{at } x_3 = l, \tag{2.19}$$

where $p_0 = P/A$ and which with $(2.6)_1$ and

$$K_{LM} = \tau_{LM}^{(1)} + \tau_{LM}^{(2)}, \tag{2.20}$$

enables us to write

$$K_{33} = -p_0,\ \mathscr{D}_{33} = rp_0,\ \text{at } x_3 = l. \tag{2.21}$$

Now, the substitution of (2.14) into (2.18) and (2.21) yields

$$A = -\frac{\gamma g + \kappa p_0 e^{-\alpha l}}{1 + e^{-2\alpha l}},\ D = -\frac{\gamma g - \kappa p_0 e^{\alpha l}}{1 + e^{2\alpha l}},$$

$$B = -(\rho g l + p_0)/\hat{c}_5,\ C = -(\hat{\beta}_6/\hat{c}_5)\gamma g, \tag{2.22}$$

where

$$\kappa = (\alpha/\hat{a}_2)(r + \hat{\beta}_6/\hat{c}_5). \tag{2.23}$$

The substitution of (2.22), with (2.23), in (2.14) yields the solution. The substitution of (2.14), with (2.22) and (2.23), into (2.1)–(2.5) yields all the stresses, relative stresses and interaction forces, which we do not bother to write. However, in the absence of g for the actual stresses τ_{33}^f in the fibers and τ_{33}^m in the matrix at the support $x_3 = 0$, we obtain

$$\tau_{33}^m = \frac{-p_0}{1 + r}\frac{A}{A^m}\left[r + \frac{\hat{\beta}_6}{\hat{c}_5} + \alpha\left(\frac{\hat{\beta}_6^2}{\hat{c}_5} - \hat{b}_5\right)\frac{\kappa}{\cosh \alpha l}\right],$$

$$\tau_{33}^f = \frac{-p_0}{1 + r}\frac{A}{A^f}\left[1 - \frac{\hat{\beta}_6}{\hat{c}_5} - \alpha\left(\frac{\hat{\beta}_6^2}{\hat{c}_5} - \hat{b}_5\right)\frac{\kappa}{\cosh \alpha l}\right]. \tag{2.24}$$

In the second specific problem, we consider fiber reinforcement entering a matrix and terminating uniformly at a distance l into the matrix which continues down to a rigid support at a distance b below the junction, as shown in Fig. 2. A total tensile force P is applied to all the fibers crossing a given cross-sectional area. Since the continuum representing the matrix and the continuum representing the fibers can neither separate from nor penetrate into the single matrix continuum that abuts the composite at the junction, we must have

$$w_3^{(1)} = 0,\ w_3^{(2)} = 0,\ \text{at } x_3 = 0, \tag{2.25}$$

which is consistent with (2.17). In addition, the displacement u_3 of the center of mass of the combined composite continuum must be the same as the displacement U_3 of the isotropic single matrix continuum at the junction. Consequently, as kinematic boundary conditions at the

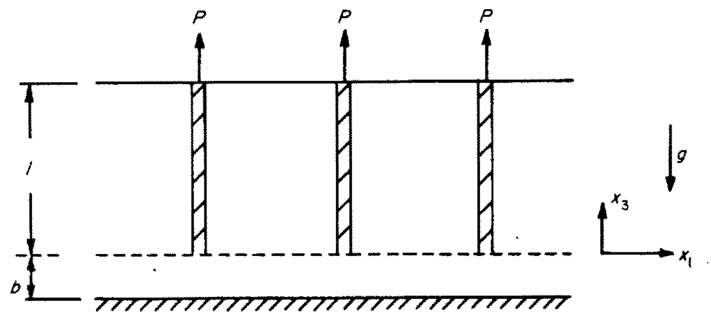

Fig. 2. Schematic diagram of loaded fiber reinforced composite with reinforcement terminating uniformly in matrix some distance before support.

junction we have

$$w_3^{(1)} = 0, \quad u_3 = U_3, \quad \text{at} \quad x_3 = 0, \tag{2.26}$$

where U_3 satisfies

$$(\lambda^m + 2\mu^m)U_{3,33} - \rho^m g = 0, \tag{2.27}$$

and the non-trivial stress components in the single matrix continuum are given by

$$T_{33}^m = (\lambda^m + 2\mu^m)U_{3,3}, \quad T_{11}^m = T_{22}^m = \lambda^m U_{3,3}. \tag{2.28}$$

In addition to the continuity of displacement at $x_3 = 0$ we have the continuity of traction, i.e.

$$K_{33} = T_{33}^m, \quad \text{at} \quad x_3 = 0. \tag{2.29}$$

Since no force is applied to the matrix at $x_3 = l$, we have

$$\tau_{33}^{(1)} = 0, \quad \tau_{33}^{(2)} = p_0, \quad \text{at} \quad x_3 = l, \tag{2.30}$$

which with $(2.6)_1$ enables us to write the boundary conditions

$$K_{33} = p_0, \quad \mathcal{D}_{33} = -rp_0, \quad \text{at} \quad x_3 = l. \tag{2.31}$$

Since the supporting surface is rigid, we have

$$U_3 = 0, \quad \text{at} \quad x_3 = -b. \tag{2.32}$$

Thus, the boundary conditions are (2.26), (2.29), (2.31) and (2.32). The solution to (2.27) may be written in the form

$$U_3 = [\rho^m g/2(\lambda^m + 2\mu^m)]x_3^2 + Ex_3 + F. \tag{2.33}$$

Now, the substitution of (2.14) and (2.33) into (2.26), (2.29), (2.31) and (2.32) yields

$$A = -\frac{\gamma g + \kappa p_0 e^{-\alpha l}}{1 + e^{-2\alpha l}}, \quad D = -\frac{\gamma g - \kappa p_0 e^{\alpha l}}{1 + e^{2\alpha l}}$$

$$B = \frac{1}{\hat{c}^5}(p_0 - \rho g l), \quad C = -\frac{\hat{\beta}_6}{\hat{c}^5}\gamma g + \frac{\left(p_0 - \rho g l - \frac{1}{2}\rho^m g b\right)b}{\lambda^m + 2\mu^m}$$

$$E = \frac{(p_0 - \rho g l)}{\lambda^m + 2\mu^m}, \quad F = \frac{\left(p_0 - \rho g l - \frac{1}{2}\rho^m g b\right)b}{\lambda^m + 2\mu^m}, \tag{2.34}$$

which when substituted in (2.14) and (2.33), respectively, yields the solution. The substitution of

(2.14) and (2.33) and (2.34) into (2.1)–(2.5) and (2.28), respectively, yields all the stresses, relative stresses and interaction forces, which we do not bother to write. However, in the absence of g, for the actual stresses τ^f_{33} in the fibers and τ^m_{33} in the matrix at the junction at $x_3 = 0$, we obtain

$$\tau^m_{33} = \frac{p_0}{1+r}\frac{A}{A^m}\left[r + \frac{\hat{\beta}_6}{\hat{c}_5} - \alpha\left(\hat{b}_5 - \frac{\hat{\beta}_6^2}{\hat{c}_5}\right)\frac{\kappa}{\cosh \alpha l}\right],$$

$$\tau^f_{33} = \frac{p_0}{1+r}\frac{A}{A^f}\left[1 - \frac{\hat{\beta}_6}{\hat{c}_5} + \alpha\left(\hat{b}_5 - \frac{\hat{\beta}_6^2}{\hat{c}_5}\right)\frac{\kappa}{\cosh \alpha l}\right]. \tag{2.35}$$

Using a highly simplified model, the results of this analysis are presented in the Appendix in terms of the known material constants of the fiber reinforcement and matrix and the relative geometry.

3. SURFACE WAVES

In this section we consider surface waves propagating along the free-surface of a semi-infinite fiber reinforced composite material. Only the case of straight-crested surface waves propagating in the direction of the fiber reinforcement, which runs parallel to the free-surface, is considered because the method of analysis is essentially the same for surface waves propagating normal to the direction of the fiber reinforcement, or in any other direction for that matter, and calculations based on the analysis are not performed because the material constants are not presently known for any two-constituent composite material. However, in the next section calculations are performed for surface waves propagating both in and normal to the direction of the fiber reinforcement using a highly simplified model of the material, and it is indicated that certain of the surface waves allowed by the simplified model will not be permitted by the more general model considered in this section.

A schematic diagram of the free-surface of the two-constituent composite along with the associated coordinate system is shown in Fig. 3. Since we consider straight-crested surface waves propagating in the direction of the fiber reinforcement, which is in the x_3-direction, the solution functions are independent of x_2. Furthermore, an examination of the equations for the two-constituent transversely isotropic composite with x_3 the preferred direction[5] presented in Ref. [1] reveals that for x_2-independence u_2 and w_2 uncouple from u_1, w_1, u_3 and w_3. Hence, under these circumstances u_2 and w_2 may be taken to vanish and the non-trivial differential equations may be written in the form

$$(c_1 + 2c_3)u_{1,11} + c_4 u_{1,33} + (\beta_1 + \beta_2)w^{(1)}_{1,11} + \frac{1}{2}\beta_4 w^{(1)}_{1,33} + (c_2 + c_4)u_{3,13} + \left(\frac{1}{2}\beta_4 + \beta_3\right)w^{(1)}_{3,13} = \rho \ddot{u}_1, \tag{3.1}$$

$$(\beta_1 + \beta_2)u_{1,11} + \frac{1}{2}\beta_4 u_{1,33} + (2b_1 + b_2)w^{(1)}_{1,11} + \left(b_4 + \frac{1}{2}b_7\right)w^{(1)}_{1,33}$$
$$+ \left(\frac{1}{2}\beta_4 + \beta_5\right)u_{3,13} + \left(b_3 + b_4 - \frac{1}{2}b_7\right)w^{(1)}_{3,13} - a_1 w_1^{(1)} = r\rho \ddot{w}_1^{(1)}, \tag{3.2}$$

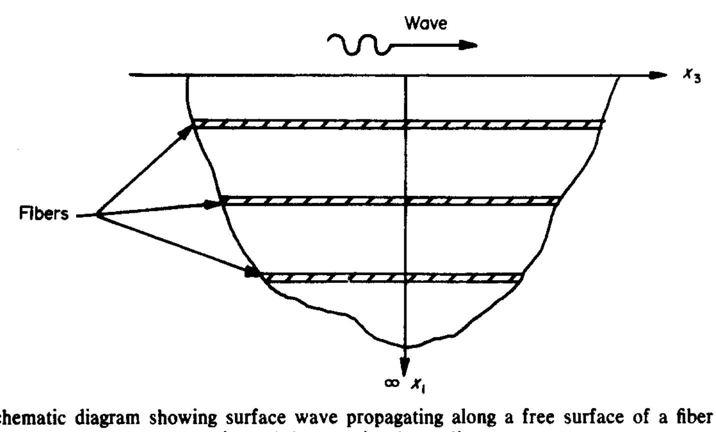

Fig. 3. Schematic diagram showing surface wave propagating along a free surface of a fiber reinforced composite and the associated coordinate system.

$$(c_2+c_4)u_{1,13}+\left(\frac{1}{2}\beta_4+\beta_5\right)w^{(1)}_{1,13}+c_4u_{3,11}+c_5u_{3,33}+\frac{1}{2}\beta_4w^{(1)}_{3,11}+\beta_6w^{(1)}_{3,33}=\rho\ddot{u}_3, \tag{3.3}$$

$$\left(\frac{1}{2}\beta_4+\beta_3\right)u_{1,13}+\left(b_3+b_4-\frac{1}{2}b_7\right)w^{(1)}_{1,13}+\frac{1}{2}\beta_4u_{3,11}$$
$$+\beta_6u_{3,33}+\left(b_4+\frac{1}{2}b_7\right)w^{(1)}_{3,11}+b_5w^{(1)}_{3,33}-a_2w_3{}^{(1)}=r\rho\ddot{w}_3{}^{(1)}, \tag{3.4}$$

and the non-trivial boundary conditions may be written in the form

$$K_{11}=K_{13}=\mathscr{D}_{11}=\mathscr{D}_{13}=0 \quad \text{at} \quad x_1=0, \tag{3.5}$$

where

$$K_{11}=(c_1+2c_3)u_{1,1}+(\beta_1+\beta_2)w^{(1)}_{1,1}+(c_1+c_2)u_{3,3}+(\beta_1+\beta_3)w^{(1)}_{3,3}, \tag{3.6}$$

$$\mathscr{D}_{11}=(\beta_1+\beta_2)u_{1,1}+(2b_1+b_2)w^{(1)}_{1,1}+(\beta_1+\beta_5)u_{3,3}+(b_2+b_3)w^{(1)}_{3,3}, \tag{3.7}$$

$$K_{13}=c_4(u_{3,1}+u_{1,3})+\frac{1}{2}\beta_4(w^{(1)}_{3,1}+w^{(1)}_{1,3}), \tag{3.8}$$

$$\mathscr{D}_{13}=\frac{1}{2}\beta_4(u_{1,3}+u_{3,1})+\left(b_4-\frac{1}{2}b_7\right)w^{(1)}_{1,3}+\left(b_4+\frac{1}{2}b_7\right)w^{(1)}_{3,1}, \tag{3.9}$$

and everything vanishes as $x_1\to\infty$.

As a solution of the differential equations we take

$$u_\alpha=A_\alpha e^{i(\eta x_1+\xi x_3-\omega t)}, \quad w_\alpha{}^{(1)}=B_\alpha e^{i(\eta x_1+\xi x_3-\omega t)}, \quad \alpha=1,3, \tag{3.10}$$

which satisfies (3.1)–(3.4) provided

$$\begin{aligned}
&(Q_{11}-\rho\omega^2)A_1+Q_{12}B_1+Q_{13}A_3+Q_{14}B_3=0,\\
&Q_{21}A_1+(Q_{22}-r\rho\omega^2)B_1+Q_{23}A_3+Q_{24}B_3=0,\\
&Q_{31}A_1+Q_{32}B_1+(Q_{33}-\rho\omega^2)A_3+Q_{34}B_3=0,\\
&Q_{41}A_1+Q_{42}B_1+Q_{43}A_3+(Q_{44}-r\rho\omega^2)B_3=0,
\end{aligned} \tag{3.11}$$

where

$$Q_{11}=(c_1+2c_3)\eta^2+c_4\xi^2, \quad Q_{12}=Q_{21}=(\beta_1+\beta_2)\eta^2+\frac{1}{2}\beta_4\xi^2,$$

$$Q_{13}=Q_{31}=(c_2+c_4)\eta\xi, \quad Q_{14}=Q_{41}=\left(\frac{1}{2}\beta_4+\beta_3\right)\eta\xi,$$

$$Q_{22}=(2b_1+b_2)\eta^2+\left(b_4+\frac{1}{2}b_7\right)\xi^2+a_1, \quad Q_{23}=Q_{32}=\left(\frac{1}{2}\beta_4+\beta_5\right)\xi\eta,$$

$$Q_{24}=Q_{42}=\left(b_3+b_4-\frac{1}{2}b_7\right)\xi\eta, \quad Q_{33}=c_4\xi^2+c_5\eta^2,$$

$$Q_{34}=Q_{43}=\frac{1}{2}\beta_4\eta^2+\beta_6\xi^2, \quad Q_{44}=\left(b_4+\frac{1}{2}b_7\right)\eta^2+b_5\xi^2+a_2. \tag{3.12}$$

Equations (3.11) constitute a system of four linear homogeneous algebraic equations in A_1, B_1, A_3 and B_3, which yields non-trivial solutions when the determinant of the coefficients of A_1, B_1, A_3 and B_3 vanishes, i.e. when

$$\begin{vmatrix}
(Q_{11}-\rho\omega^2) & Q_{12} & Q_{13} & Q_{14}\\
Q_{21} & (Q_{22}-r\rho\omega^2) & Q_{23} & Q_{24}\\
Q_{31} & Q_{32} & (Q_{33}-\rho\omega^2) & Q_{34}\\
Q_{41} & Q_{42} & Q_{43} & (Q_{44}-r\rho\omega^2)
\end{vmatrix}=0. \tag{3.13}$$

Equation (3.13) is a quartic in ω^2, ξ^2 and η^2 and, hence, for a given ω and ξ there are four in general complex η^2. Since the solution functions must vanish as $x_1 \to \infty$, only those $\eta^{(n)}$ ($n = 1, 2, 3, 4$) with positive imaginary part are admissible. For a given $\eta^{(n)}$, three of the four eqns in (3.11) yield amplitude ratios, which we denote

$$A_1^{(n)}: B_1^{(n)}: A_3^{(n)}: B_3^{(n)}. \tag{3.14}$$

All four admissible solutions at a given ξ and ω are required in order to satisfy the four boundary conditions in (3.5). Consequently, we take

$$u_\alpha = \sum_{n=1}^{4} C^{(n)} A_\alpha^{(n)} e^{i\eta^{(n)}x_1} e^{i(\xi x_3 - \omega t)},$$

$$w_\alpha^{(1)} = \sum_{n=1}^{4} C^{(n)} B_\alpha^{(n)} e^{i\eta^{(n)}x_1} e^{i(\xi x_3 - \omega t)}, \quad \alpha = 1,3, \tag{3.15}$$

which satisfy (3.5) provided

$$\sum_{n=1}^{4} C^{(n)} L_\gamma^{(n)} = 0, \quad \gamma = 1,2,3,4, \tag{3.16}$$

where

$$L_1^{(n)} = (c_1 + 2c_3)\eta^{(n)} A_1^{(n)} + (\beta_1 + \beta_2)\eta^{(n)} B_1^{(n)} + (c_1 + c_2)\xi A_3^{(n)} + (\beta_1 + \beta_3)\xi B_3^{(n)},$$

$$L_2^{(n)} = (\beta_1 + \beta_2)\eta^{(n)} A_1^{(n)} + (2b_1 + b_2)\eta^{(n)} B_1^{(n)} + (\beta_1 + \beta_5)\xi A_3^{(n)} + (b_2 + b_3)\xi B_3^{(n)}$$

$$L_3^{(n)} = c_4 \xi A_1^{(n)} + \frac{1}{2}\beta_4 \xi B_1^{(n)} + c_4 \eta^{(n)} A_3^{(n)} + \frac{1}{2}\beta_4 \eta^{(n)} B_3^{(n)},$$

$$L_4^{(n)} = \frac{1}{2}\beta_4 \xi A_1^{(n)} + \left(b_4 - \frac{1}{2}b_7\right)\xi B_1^{(n)} + \frac{1}{2}\beta_4 \eta^{(n)} A_3^{(n)} + \left(b_4 + \frac{1}{2}b_7\right)\eta^{(n)} B_3^{(n)}. \tag{3.17}$$

Equations (3.16) constitute four linear homogeneous algebraic equations in the $C^{(n)}$, and for a non-trivial solution the determinant of the coefficients must vanish, i.e.

$$\begin{vmatrix} L_1^{(1)} & L_1^{(2)} & L_1^{(3)} & L_1^{(4)} \\ L_2^{(1)} & L_2^{(2)} & L_2^{(3)} & L_2^{(4)} \\ L_3^{(1)} & L_3^{(2)} & L_3^{(3)} & L_3^{(4)} \\ L_4^{(1)} & L_4^{(2)} & L_4^{(3)} & L_4^{(4)} \end{vmatrix} = 0. \tag{3.18}$$

If the constants are known, a calculation proceeds by selecting values for ξ and ω, which enables the determination of the four $\eta^{(n)}$ from (3.13) and the attendant amplitude ratios from (3.11). Then everything in (3.18) is known and either it is satisfied or it is not. If (3.18) is not satisfied, change either ξ or ω and repeat the entire calculation until (3.18) is satisfied. Experience with surface waves indicates that there will be one ω at a given ξ that satisfies all conditions, i.e. (3.13) and (3.18). However, since the constants are not known for any two-constituent composite material, a calculation cannot be performed. In the next section calculations are performed for a highly simplified version of the model for which the constants can be estimated from the known constants of the two constituents in the composite.

4. SURFACE WAVES AND THE SIMPLIFIED MODEL

In this section we simplify the equations for the two-constituent transversely isotropic composite by reducing the model sufficiently that the material constants of the composite can be estimated from the known constants of the individual constituents of the composite while still retaining certain of the essential characteristics of the composite. Since in the simplified model the constants are essentially known, calculations can be and, indeed, are performed for surface waves propagating in two-constituent transversely isotropic composite materials. The simplified model we are considering is for a fiber reinforced composite material consisting of an elastic matrix containing uniformly distributed continuous fibers extending in the x_3-direction, in which the fibers occupy a small fraction of the total composite volume. On account of the latter condition in the reduced model it is assumed that the stresses in the matrix are related to the strains in the matrix by the constitutive relations of linear isotropic elasticity and are independent of the strains in the fibers. Similarly, the stresses in the fibers are assumed to be independent of the strains in the matrix. It is further assumed that all stress components in the long narrow fibers vanish save the axial stress $\tau_{33}^{(2)}$, which may then be written as a function of the axial strain in the fibers only. Although this latter assumption seems questionable to us in general we make it anyway. Then the only interaction between the matrix and the fibers remaining is the volumetric interaction term ${}^LF_M^{12}$, which from (2.5) and (2.7) takes the form[5]

$$ {}^LF_P^{12} = -(1+r)^{-1}a_1 w_P^{(1)}, \quad {}^LF_3^{12} = -(1+r)^{-1}a_2 w_3^{(1)}, \quad P = 1,2. \tag{4.1} $$

At this point it should be noted that the aforementioned assumptions make the reduced model for the linear case identical with that of Martin, Bedford and Stern[4].

On account of the assumptions made in the simplified model of the two-constituent composite, it is advantageous to write the equations in terms of the infinitesimal displacement fields of each constituent instead of the center of mass and relative displacement fields. To this end we write[6]

$$ u_K^{(1)} = u_K + w_K^{(1)}, \quad u_K^{(2)} = u_K + w_K^{(2)}, \tag{4.2} $$

which with the three-dimensional version of (2.17)

$$ r w_K^{(1)} = -w_K^{(2)}, \tag{4.3} $$

enables us to write

$$ u_K = (1+r)^{-1}(r u_K^{(1)} + u_K^{(2)}), \quad w_K^{(1)} = (1+r)^{-1}(u_K^{(1)} - u_K^{(2)}). \tag{4.4} $$

From $(2.6)_1$ and (2.20) we obtain

$$ \tau_{LM}^{(1)} = (1+r)^{-1}(rK_{LM} + \mathcal{D}_{LM}), \quad \tau_{LM}^{(2)} = (1+r)^{-1}(K_{LM} - \mathcal{D}_{LM}). \tag{4.5} $$

Substituting from (4.4) into the constitutive equations for the two-constituent isotropic composite[7] and then into (4.5), and introducing the aforementioned assumptions of independence of the stress in the matrix continuum on the strain in the fiber continuum and vice versa, which introduces relations among the more generally defined material constants, and the further assumption of uniaxial stress in the fiber continuum, we obtain

$$ \begin{aligned} \tau_{LM}^{(1)} &= \lambda^{(1)} u_{K,K}^{(1)} \delta_{LM} + \mu^{(1)}(u_{L,M}^{(1)} + u_{M,L}^{(1)}), \\ \tau_{LM}^{(2)} &= E^{(2)} u_{3,3}^{(2)} \delta_{LM}, \end{aligned} \tag{4.6} $$

where $\lambda^{(1)}$ and $\mu^{(1)}$ are the Lamé constants of the matrix continuum and $E^{(2)}$ is Young's modulus for the fiber continuum, which are related to the respective constants in the matrix and fibers by

$$ \lambda^{(1)} = \lambda^m A^m/A, \quad \mu^{(1)} = \mu^m A^m/A, \quad E^{(2)} = E^f A^f/A. \tag{4.7} $$

Since the fiber reinforcement in this simplified model is restricted to occupy a small fraction of the total composite volume, (4.7) enables us to write

$$ \lambda^{(1)} = \lambda^m, \quad \mu^{(1)} = \mu^m, \quad E^{(2)} = E^f N S^f, \tag{4.8} $$

where S^f is the cross-sectional area of each fiber and N is the number of fibers per unit area. In order to complete the constitutive equations for the reduced model we substitute from (4.4) into (4.1) to obtain

$$ {}^LF_P^{12} = -\tilde{a}_1(u_P^{(1)} - u_P^{(2)}), \quad {}^LF_3^{12} = -\tilde{a}_2(u_3^{(1)} - u_3^{(2)}), \tag{4.9} $$

where

$$\tilde{a}_1 = a_1/(1+r)^2, \quad \tilde{a}_2 = a_2/(1+r)^2. \tag{4.10}$$

The stress and relative stress equations of motion in the absence of body and relative body forces take the respective forms

$$K_{LM,L} = \rho \ddot{u}_M, \quad \mathcal{D}_{LM,L} + \mathcal{F}_M = r\rho \ddot{w}_M^{(1)}, \tag{4.11}$$

which with (2.6), (2.7), (2.20) and (4.4) enables us to write

$$\tau^{(1)}_{LM,L} + {}^{L}F_M{}^{12} = \rho^{(1)} \ddot{u}_M{}^{(1)}, \tag{4.12}$$

$$\tau^{(2)}_{LM,L} - {}^{L}F_M{}^{12} = \rho^{(2)} \ddot{u}_M{}^{(2)}, \tag{4.13}$$

the latter of which, on account of the assumption of uniaxial stress in the fibers yields

$$\tau^{(2)}_{33,3}\delta_{LM} - {}^{L}F_M{}^{12} = \rho^{(2)} \ddot{u}_M{}^{(2)}. \tag{4.14}$$

Substituting from (4.6) and (4.9) into (4.12) and (4.14) and employing (4.8), we obtain

$$(\lambda^m + \mu^m) u^{(1)}_{K,KL} + \mu^m u^{(1)}_{L,KK} + \tilde{a}_1(u_L{}^{(2)} - u_L{}^{(1)}) + (\tilde{a}_2 - \tilde{a}_1)(u_L{}^{(2)} - u_L{}^{(1)})\delta_{3L} = \rho^{(1)}\ddot{u}_L{}^{(1)}, \tag{4.15}$$

$$E^{(2)} u^{(2)}_{3,33}\delta_{L3} + \tilde{a}_1(u_L{}^{(1)} - u_L{}^{(2)}) + (\tilde{a}_2 - \tilde{a}_1)(u_L{}^{(1)} - u_L{}^{(2)})\delta_{L3} = \rho^{(2)}\ddot{u}_L{}^{(2)}, \tag{4.16}$$

which are the displacement equations of motion of this highly simplified model of the two-constituent composite material. Equations (4.15) and (4.16) are identical with the equations of Martin, Bedford and Stern[4], who have provided an approximate procedure for estimating the constant $\tilde{a}_2$ in terms of the known constants of the individual constituents in the composite and the geometry. Their analysis provides the result

$$\tilde{a}_2 = \frac{(\mu^m/h^2)(1-\delta^2/h^2)^2}{\left[\frac{1}{2}\log(h/\delta) + \frac{1}{2}\delta^2/h^2 - \frac{1}{8}\delta^4/h^4 - \frac{3}{8}\right] + \frac{1}{8}(\mu^m/\mu^f)(1-\delta^2/h^2)^2}, \tag{4.17}$$

where

$$h^2 = (\sqrt{(3)}/2\pi)s^2, \tag{4.18}$$

and δ is the fiber radius, h is the radius of a cylinder, each of which encloses a single fiber in the hexagonal array and abuts all adjacent cylinders, s is the fiber spacing and μ^m and μ^f are the shear moduli of the matrix and fiber material, respectively. However, the constant $\tilde{a}_1$ still remains undetermined in this simplified description and it does not appear to be possible to estimate it in some approximate manner because an appropriate problem yielding a simple solution of a full linearly elastic boundary value problem cannot be found. Nevertheless, since we are introducing this highly limited simplified description in order to obtain some numerical results, we take the rather arbitrary course of assuming that $\tilde{a}_1 = \tilde{a}_2$. Under these circumstances all material constants in the equations are known and calculations can readily be performed.

As in the case of the more general model, in the consideration of straight-crested surface waves propagating in the direction of the fiber reinforcement (x_3-direction), the solution functions are independent of x_2 and $u_2{}^{(1)}$ and $u_2{}^{(2)}$ vanish. Under these circumstances we have the differential eqns (4.15) and (4.16), with x_2-independence understood and $u_2{}^{(1)} = u_2{}^{(2)} = 0$, along with the non-trivial boundary conditions

$$\tau^{(1)}_{11} = \tau^{(1)}_{13} = \tau^{(2)}_{11} = \tau^{(2)}_{13} = 0, \quad \text{at} \quad x_1 = 0. \tag{4.19}$$

Since $\tau^{(2)}_{11}$ and $\tau^{(2)}_{13}$ vanish identically in the simplified model, we have only the two non-trivial boundary conditions

$$\tau^{(1)}_{11} = \tau^{(1)}_{13} = 0, \quad \text{at} \quad x_1 = 0. \tag{4.20}$$

It is our belief that the foregoing is essentially the reason that the surface wave solutions in the simplified model yield upper (optical type) surface wave branches, which we feel will not be in conformity with results that would be obtained from the more general model.

As a solution of the differential equations we take

$$u_\alpha^{(1)} = A_\alpha e^{i(\eta x_1 + \xi x_3 - \omega t)}, \quad u_\alpha^{(2)} = B_\alpha e^{i(\eta x_1 + \xi x_3 - \omega t)}. \quad \alpha = 1,3, \tag{4.21}$$

which satisfies (4.15) and (4.16) provided

$$\begin{aligned} &(P_{11} - \rho^{(1)}\omega^2)A_1 - \tilde{a}_1 B_1 - P_{13}A_3 = 0, \\ &-\nu A_1 \times (\tilde{a}_1 - \rho^{(2)}\omega^2)B_1 = 0, \\ &P_{31}A_1 + (P_{33} - \rho^{(1)}\omega^2)A_3 - \tilde{a}_2 B_3 = 0, \\ &-\tilde{a}_2 A_3 + (P_{44} - \rho^{(2)}\omega^2)B_3 = 0, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} P_{11} &= (\lambda^m + 2\mu^m)\eta^2 + \mu^m\xi^2 + \tilde{a}_1, \quad P_{13} = P_{31} = (\lambda^m + \mu^m)\eta\xi, \\ P_{33} &= +\mu^m\eta^2 + (\lambda^m + 2\mu^m)\xi^2 + \tilde{a}_2, \quad P_{44} = E^{(2)}\xi^2 + \tilde{a}_2. \end{aligned} \tag{4.23}$$

Equations (4.22) constitute a system of linear, homogeneous algebraic equations in A_1, B_1, A_3 and B_3, which yields non-trivial solutions when the determinant of the coefficients of A_1, A_3, B_1 and B_3 vanishes, i.e. when

$$\begin{vmatrix} P_{11} - \rho^{(1)}\omega^2 & -\tilde{a}_1 & P_{13} & 0 \\ -\tilde{a}_1 & \tilde{a}_1 - \rho^{(2)}\omega^2 & 0 & 0 \\ P_{31} & 0 & P_{33} - \rho^{(1)}\omega^2 & -\tilde{a}_2 \\ 0 & 0 & -\tilde{a}_2 & P_{44} - \rho^{(2)}\omega^2 \end{vmatrix} = 0. \tag{4.24}$$

Equation (4.24) is quadratic in η^2, cubic in ξ^2 and quartic in ω^2, and, hence, for a given ω and ξ there are two in general complex η^2. Since the solution functions must vanish as $x_1 \to \infty$, only those $\eta^{(n)}(n = 1, 2)$ with positive imaginary part are admissible. For a given $\eta^{(n)}$ three of the four equations in (4.22) yield amplitude ratios, which we denote

$$A_1^{(n)} : B_1^{(n)} : A_3^{(n)} : B_3^{(n)}. \tag{4.25}$$

Since only two non-trivial boundary conditions remain in (4.20), the two admissible solutions are adequate and we take

$$\begin{aligned} u_\alpha^{(1)} &= \sum_{n=1}^{2} C^{(n)} A_\alpha^{(n)} e^{i\eta^{(n)}x_1} e^{i(\xi x_3 - \omega t)}, \\ u_\alpha^{(2)} &= \sum_{n=1}^{2} C^{(n)} B_\alpha^{(n)} e^{1\eta^{(n)}x_1} e^{i(\xi x_3 - \omega t)}, \end{aligned} \tag{4.26}$$

which satisfy (4.20) provided

$$\begin{aligned} &\sum_{n=1}^{2} C^{(n)}[i\eta^{(n)}(\lambda^m + 2\mu^m)A_1^{(n)} + i\lambda^m\xi A_3^{(n)}] = 0, \\ &\sum_{n=1}^{2} C^{(n)}[i\xi A_1^{(n)} + i\eta^{(n)}A_3^{(n)}] = 0. \end{aligned} \tag{4.27}$$

At this point it should be noted that on account of the simplified model eqns (4.27) do not contain the $B_\alpha^{(n)}$. Equations (4.27) constitute two linear homogeneous algebraic equations in $C^{(1)}$ and $C^{(2)}$, and for a non-trivial solution the determinant of the coefficients must vanish, i.e.

$$\begin{vmatrix} (\lambda^m + 2\mu^m)\eta^{(1)}A_1^{(1)} + \lambda^m\xi A_3^{(1)} & (\lambda^m + 2\mu^m)\eta^{(2)}A_1^{(2)} + \lambda^m\xi A_3^{(2)} \\ \xi A_1^{(1)} + \eta^{(1)}A_3^{(1)} & \xi A_1^{(2)} + \eta^{(2)}A_3^{(2)} \end{vmatrix} = 0. \tag{4.28}$$

Equation (4.28) is a complex algebraic equation, both the real and imaginary parts of which must vanish simultaneously. Solutions, i.e. values of ω and ξ satisfying (4.24) and (4.28) are found numerically in a manner similar to that discussed in the paragraph following eqn (3.18).

Calculations have been performed for a set of material parameters corresponding to a glass fiber reinforced phenolic resin[4], the relevant constants of which are

$$\rho^m = 0.00013\,\text{lb-sec}^2/\text{in}^4, \quad E^f = 12.4\times 10^6\,\text{lb/in}^2,$$
$$\rho^f = 0.00026\,\text{lb-sec}^2/\text{in}^4, \quad \mu^f = 10.2\times 10^6\,\text{lb/in}^2,$$
$$\lambda^m = 0.86\times 10^6\,\text{lb/in}^2, \quad \mu^m = 0.37\times 10^6\,\text{lb/in}^2, \tag{4.29}$$

for a fiber diameter of 0.01 in. for the volume percentage of reinforcement of 5.67%, which corresponds to $s = 0.04$ in. The results of the calculations are plotted in Fig. 4, which indicates the existence of an upper (optical type) surface wave branch in addition to the lower (acoustic type) branch. As already noted, we do not believe that the upper surface wave branch actually exists, but that its existence is a consequence of the reduced coupling in the simplified model. A similar analysis has been performed for straight-crested surface waves propagating normal to the direction of the fiber reinforcement and the results of calculations based on this analysis are plotted in Fig. 5. Again the results indicate the existence of an upper as well as a lower branch

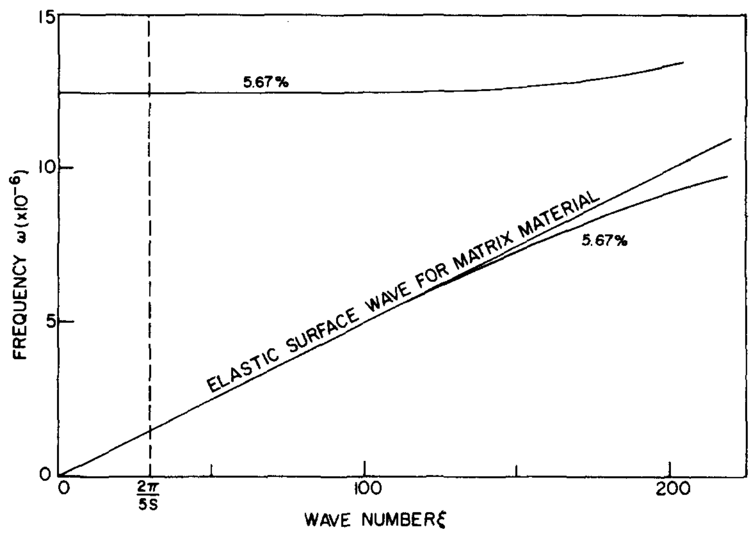

Fig. 4. Dispersion curves for surface waves propagating in the direction of the fiber reinforcement.

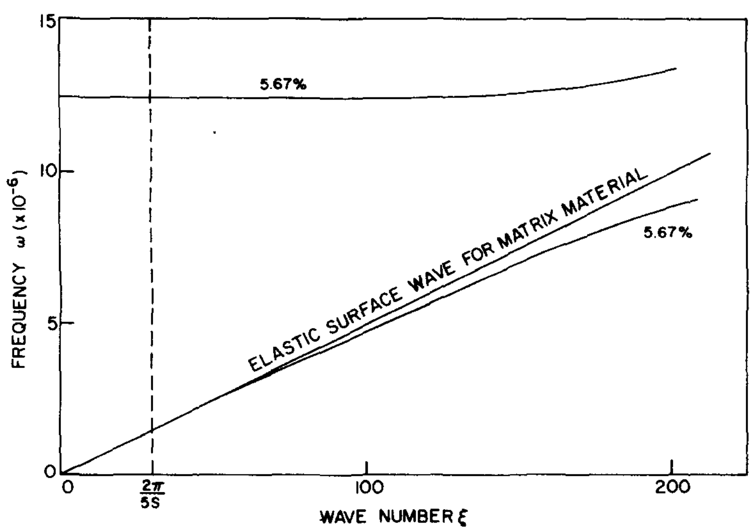

Fig. 5. Dispersion curves for surface waves propagating normal to the direction of the fiber reinforcement.

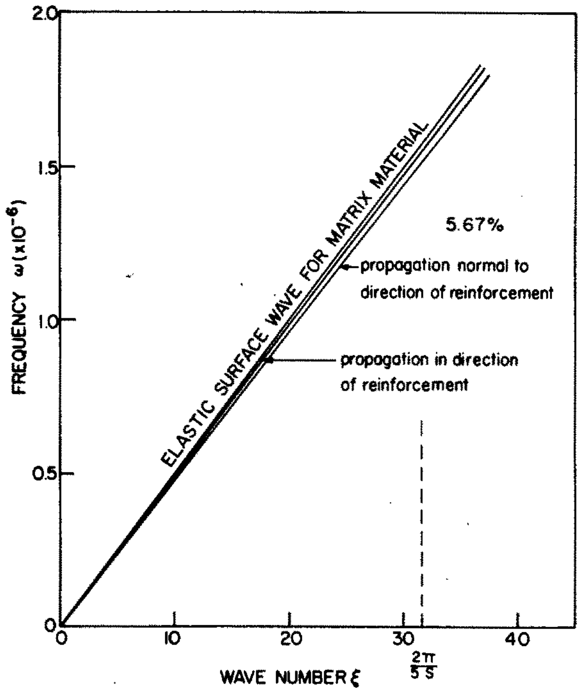

Fig. 6. Acoustic type dispersion curves for surface waves propagating both in and normal to the direction of the fiber reinforcement.

and we do not believe that the upper branch actually exists for the aforementioned reasons. In Figs. 4 and 5 we have drawn vertical lines which correspond to a wavelength five times the spacing of the fiber reinforcement. We do not believe the curves to be valid much beyond these vertical lines because of the nature of the model of the composite we have employed, and we draw them considerably beyond their range of validity simply to indicate the calculated behavior. The important curves in Figs. 4 and 5 are the lower acoustic type branches, both of which are drawn to a larger scale in Fig. 6. Note the difference in dispersion for the two directions of propagation considered. This very precise dispersion property of surface waves could well be used as a means of non-destructively evaluating the distribution of the fiber reinforcement in and the integrity of the bonding to the matrix.

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5. Ref. [1], eqns (10.8)–(10.10).
6. Ref. [1], eqns (2.4) and (9.1).
7. Ref. [1], eqns (10.4)–(10.6).

APPENDIX

In this Appendix we present the solution to the second problem treated in Section 2, i.e. the one associated with Fig. 2, using the reduced equations presented in Section 4 and omitting the influence of gravity. In this case the differential eqns (4.15) and (4.16) with $L = 3$ and the constitutive eqns (4.6) with $L = M = 3$ and $(4.9)_2$, all in the absence of any x_1-x_2-dependence and inertial terms, respectively replace eqns (2.12), (2.13), (2.2), (2.4) and (2.5). Equations (2.27), (2.28), (2.30) and (2.32) remain unchanged and (2.26) and (2.29) are replaced by

$$u_3^{(1)} = u_3^{(2)}, \quad u_3^{(1)} = U_3 \quad \text{at} \quad x_3 = 0, \tag{A1}$$

$$\tau_{33}^{(1)} + \tau_{33}^{(2)} = T_{33}^m, \quad \text{at} \quad x_3 = 0. \tag{A2}$$

The solution functions given in (2.14) and (2.33) are replaced by

$$u_3^{(1)} = C_1 + C_2 x_3 + C_3 e^{\alpha x_3} + C_4 e^{-\alpha x_3}, \quad U_3 = C_5 + C_6 x_3,$$

$$u_3^{(2)} = C_1 + C_2 x_3 + \left[1 - \alpha^2 \frac{(\lambda^m + 2\mu^m)}{\bar{a}_2}\right](C_3 e^{\alpha x_3} + C_4 e^{-\alpha x_3}), \tag{A3}$$

where

$$\alpha = [\bar{a}_2(\lambda^m + 2\mu^m + E^{(2)})/E^{(2)}(\lambda^m + 2\mu^m)]^{1/2}. \tag{A4}$$

Substituting from (A3) into (A1), (A2), (2.30) and (2.32), we obtain

$$C_1 = \frac{p_0 b}{\lambda^m + 2\mu^m}, \quad C_2 = \alpha\beta p_0, \quad C_3 = \frac{-p_0\beta}{2\cosh \alpha l},$$

$$C_4 = \frac{p_0\beta}{2\cosh \alpha l}, \quad C_5 = \frac{p_0 b}{\lambda^m + 2\mu^m}, \quad C_6 = \frac{p_0}{\lambda^m + 2\mu^m}, \tag{A5}$$

where

$$\beta = 1/\alpha(\lambda^m + 2\mu^m + E^{(2)}). \tag{A6}$$

The actual stresses τ_{33}^f in the fibers and τ_{33}^m in the matrix at the junction are given by

$$\tau_{33}^f = \frac{p_0^f}{\lambda^m + 2\mu^m + E^{(2)}}\left[E^{(2)} + \frac{\lambda^m + 2\mu^m}{\cosh \alpha l}\right],$$

$$\tau_{33}^m = \frac{(p_0^f A^f/A)(\lambda^m + 2\mu^m)}{\lambda^m + 2\mu^m + E^{(2)}}\left[1 - \frac{1}{\cosh \alpha l}\right], \tag{A7}$$

where

$$p_0^f = p_0 A/A^f, \quad E^{(2)} = E^f A^f/A, \tag{A8}$$

and p_0^f denotes the actual stress in the fibers before they enter the matrix.